Abstract Algebra. 29-07-2020.

B.Sc. part-I (H), paper-I

Q.1 The intersection of any two subgroups of a group G is a subgroup of G.

Proof: Let H_1 and H_2 be two subgroups of a group G.

We have to prove that $H_1 \cap H_2$ is a subgroup of G.

For this, we have to show that $a, b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$.

Let $a, b \in H_1 \cap H_2 \Rightarrow a, b \in H_1$ and $a, b \in H_2$

Now, since H_1 and H_2 are subgroups, then

$$a, b \in H_1 \Rightarrow ab^{-1} \in H$$

and $a, b \in H_2 \Rightarrow ab^{-1} \in H_2$

$ab^{-1} \in H_1$ and $ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$ proved.

Q.2 An arbitrary intersection of subgroups of a group G is a subgroup of G.

Proof: Let H_r be the collection of subgroups for $r \in N$.

Let $H = \bigcap_{r=1}^{\infty} H_r$

We have to show that H is a subgroup of G

$$a, b \in H \Rightarrow a \in \bigcap_{r=1}^{\infty} H_r \text{ and } b \in \bigcap_{r=1}^{\infty} H_r$$

$$\Rightarrow a \in H_r, b \in H_r, \forall r \in N.$$

$$\Rightarrow ab^{-1} \in H_r, \forall r \in N$$

$$\Rightarrow ab^{-1} \in \bigcap_{r=1}^{\infty} H_r = H$$

$$\Rightarrow ab^{-1} \in H$$

Thus, we have proved that $a, b \in H \Rightarrow ab^{-1} \in H$

This declares that H is a subgroup of G.

Q.3. The union of two subgroups of a group G is a subgroup of G iff one is contained in the other.

Proof: Let H_1 and H_2 be subgroups of a group G. Let $H_1 \subset H_2$ or $H_2 \subset H_1$.

We have to show that $H_1 \cup H_2$ is a subgroup of G.

$H_1 \subset H_2 \Rightarrow H_1 \cup H_2 = H_2$

$H_1 \subset H_2 \Rightarrow H_1 \cup H_2$ is a subgroup of G.

Also, H_2 is a subgroup of G $\Rightarrow H_1 \cup H_2 = H_1$.

Again, $H_2 \subset H_1 \Rightarrow H_1 \cup H_2$ is a subgroup of G.

Also, H_1 is a subgroup of $G \Rightarrow H_1 \cup H_2$ is a subgroup of G, in both cases.

Hence, $H_1 \cup H_2$ is a subgroup

Conversely: Let us suppose that H_1 and H_2 are subgroups

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Rest part of Q.3.

of a group G such that $H_1 \cup H_2$ is a subgroup of G.

We have to show that $H_1 \subset H_2$ or $H_2 \subset H_1$

Suppose the contrary. Then $H_1 \not\subset H_2$ or $H_2 \not\subset H_1$

$H_1 \not\subset H_2 \Rightarrow \exists\, a \in H_1$ such that $a \notin H_2$

and $H_2 \not\subset H_1 \Rightarrow \exists\, b \in H_2$ such that $b \notin H_1$.

Now, $a, b \in H_1 \cup H_2$, and $H_1 \cup H_2$ is a subgroup of G.

$\Rightarrow ab \in H_1 \cup H_2$

This implies $ab \in H_1$ or $ab \in H_2$

$a \in H_1, ab \in H_1 \Rightarrow a^{-1}(ab) \in H_1$ [$\because H_1$ is a subgroup]

$\Rightarrow (a^{-1}a)b \in H_1 \Rightarrow eb \in H_1 \Rightarrow b \in H_1$

Which is a contradiction. [$\because b \notin H_1$)

$b \in H_2, ab \in H_2 \Rightarrow (ab)b^{-1} \in H_2$

$\Rightarrow a \in H_2$ [For $(ab)b^{-1} = a(bb^{-1}) = ae = a$]

Again, we get a contradiction [$\because a \notin H_2$]

Hence, our initial assumption is wrong

Consequently $H_1 \subset H_2$ or $H_2 \subset H_1$.

proved.

Q.4. The union of two subgroups is not necessarily a subgroup.

proof: Let H_1 and H_2 be two subgroups of the group $(\mathbb{Z}, +)$

where. $H_1 = (2n : n \in \mathbb{Z})$, $H_2 = (5n : n \in \mathbb{Z})$

Then, $H_1 \cup H_2 = [x : x = 2n$ or $5n$ where $n \in \mathbb{N}]$

$= [0, \pm 2, \pm 4, \pm 6, \ldots 0, \pm 5, \pm 10, \pm 15, \ldots]$

$6, 15 \in H_1 \cup H_2 \Rightarrow 6+15 = 21 \notin H_1 \cup H_2$

$\Rightarrow$ Closure property is not satisfied.

Hence, $H_1 \cup H_2$ is not a subgroups of $(\mathbb{Z}, +)$

Proved.